On normal Sylow subgroups*

Claudi Lleyda Moltó

The three Sylow Theorems

Definition 1 (Sylow 🚑-subgroup). Let **Bē** be a group of order 🚑$^{⇦}$🖧, where 🚑 is a prime number and 🚑 ∤ 🖧, such that there exists a subgroup 🌿 of **Bē** of order 🚑$^{⇦}$. Then we say that 🌿 is a Sylow 🚑-subgroup of **Bē**.

Theorem 2 (First Sylow Theorem). *Let* **Bē** *be a finite group. Then for every prime* 🚑 *dividing the order of* **Bē**. *Then there exists a Sylow* 🚑*-subgroup of* **Bē**, *and every* 🚑*-subgroup of* **Bē** *is in a Sylow* 🚑*-subgroup of* **Bē**.

Theorem 3 (Segond Sylow Theorem). *Let* **Bē** *be a finite group. Then for every prime* 🚑 *dividing the order of* **Bē**, *the Sylow* 🚑*-subgroups of* **Bē** *are conjugate.*

Theorem 4 (Third Sylow Theorem). *Let* **Bē** *be a group of order* 🚑$^{⇦}$🖧, *where* 🚑 *is a prime number and* 🚑 ∤ 🖧, *and let* ⚓$_{🚑}$ *be the number of Sylow* 🚑*-subgroups of* **Bē**. *Then*

$$⚓_{🚑} \mid 🖧 \qquad \textit{and} \qquad ⚓_{🚑} \equiv 🎂 \pmod{🚑},$$

Normal Sylow subgroups

Proposition 5. *Let* **Bē** *be a group with* ⚓$_{🚑}$ = 🎂. *Then its Sylow* 🌿*-subgroup is a normal subgroup of* **Bē**

Proof. Let 🌿 be the Sylow 🚑-subgroup of **Bē**. By the second Sylow Theorem we have that for every ✌ ∈ **Bē** we find that

$$✌🌿✌^{-🎂} = 🌿,$$

and 🌿 is a normal subgroup of **Bē**. 📷

Theorem 6. *Let* **Bē** *be a finite group such that there exist two different prime factors* 🚑 *and* ⚖ *dividing its order with* ⚓$_{🚑}$ = 🎂 *and* ⚓$_{⚖}$ = 🎂. *Then*

*🎂 = #{🎂}

the elements of the Sylow $🚑$*-subgroup commute with the elements of the Sylow* $⚖$*-subgroup of* $\mathsf{B\bar{e}}$*.*

Proof. Let $🚶$ be the group identity of $\mathsf{B\bar{e}}$ and let $🌿$ be the Sylow $🚑$-subgroup and $🔕$ be the Sylow $⚖$-subgroup of $\mathsf{B\bar{e}}$. By Lagrange's Theorem we find that

$$🌿 \cap 🔕 = \{🚶\},$$

and by the last proposition we have that $🌿$ and $🔕$ are normal subgroups of $\mathsf{B\bar{e}}$.

Then, for every $🏢 \in 🌿$ and every $🐞 \in 🔕$ we have

$$\begin{aligned} 🏢🐞🏢^{-🎂}🐞^{-🎂} &= (🏢🐞🏢^{-🎂})🐞^{-🎂} \in 🔕 \\ &= 🏢(🐞🏢^{-🎂}🐞^{-🎂}) \in 🌿 \\ &\in 🌿 \cap 🔕 = \{🚶\} \\ &= \{🚶\}, \end{aligned}$$

with which we find that $🏢🐞 = 🐞🏢$. 📷

Theorem 7. *Let* $\mathsf{B\bar{e}}$ *be a finite group, let* $🚑_{🎂}, \ldots, 🚑_{⊙}$ *be the different primes dividing the order of* $\mathsf{B\bar{e}}$ *and let* $🌿_{🎂}, \ldots, 🌿_{⊙}$ *be the Sylow subgroups of* $\mathsf{B\bar{e}}$*. Then the Sylow subgroups* $🌿_{🎂}, \ldots, 🌿_{⊙}$ *are normal if and only if*

$$\mathsf{B\bar{e}} \cong 🌿_{🎂} \times \cdots \times 🌿_{⊙}.$$

Proof. Suppose that $\mathsf{B\bar{e}} \cong 🌿_{🎂} \times \cdots \times 🌿_{⊙}$. We have

$$|\mathsf{B\bar{e}}| = |🌿_{🎂} \times \cdots \times 🌿_{⊙}| = |🌿_{🎂}| \cdots |🌿_{⊙}|,$$

which implies $⚓_{🚑_{🎂}} = 🎂, \ldots, ⚓_{🚑_{⊙}} = 🎂$, and by proposition 5 we find that the Sylow subgroups $🌿_{🎂}, \ldots, 🌿_{⊙}$ are normal.

Suppose now the Sylow subgroups $🌿_{🎂}, \ldots, 🌿_{⊙}$ are normal. By the previous theorem we have that the elements of two different groups commute. With this we find that the function

$$\begin{aligned} ☕\colon 🌿_{🎂} \times \cdots \times 🌿_{⊙} &\longrightarrow \mathsf{B\bar{e}} \\ 🏢_{🎂}, \cdots, 🏢_{⊙} &\longmapsto 🏢_{🎂} \cdots 🏢_{⊙} \end{aligned}$$

is a homomorphism. The homomorphism $☕$ is injective, since the order of a product of commuting elements with relatively primer orders is equal to the product of their orders. We also have that

$$|\mathsf{B\bar{e}}| = |🌿_{🎂} \times \cdots \times 🌿_{⊙}| = |🌿_{🎂}| \cdots |🌿_{⊙}|,$$

with which we find that $☕$ is a group isomorphism. 📷

https://kconrad.math.uconn.edu/blurbs/grouptheory/sylowapp.pdf.